



On Locally Artinian Rings

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Abstract

In this paper, a new ring is introduced and studied, which we call a locally Artinian ring and it is a generalization of an Artinian ring. Several properties of Artinian rings are extended to this new type of commutative rings. Some conditions are given under which a locally Artinian ring is Artinian. It is known that, a locally Artinian ring is locally Noetherian, but the converse is not true and an example of a locally Noetherian ring which is not locally Artinian is given. Furthermore, a necessary and sufficient condition is given for a locally Noetherian ring to be locally Artinian.

Introduction

Throughout this paper, R is a commutative ring with identity. A non empty subset S is called a multiplicatively closed set in R , if $0 \notin S$ and $a, b \in S$ implies $ab \in S$. If S is a multiplicatively closed set in R , then define a relation (\sim) on $R \times S$ as: for $(r, s), (t, u) \in R \times S$, we say $(r, s) \sim (t, u)$ if and only if there exists $v \in S$ such that $v(ur - st) = 0$, then (\sim) is an equivalence relation on $R \times S$ and if we denote the equivalence class of (r, s) by $\frac{r}{s}$ and the set of all equivalence classes by R_S , then R_S is a commutative ring with the identity $\frac{s}{s}$, where $s \in S$, under the operations, $\frac{r}{s} + \frac{t}{u} = \frac{ur+st}{su}$ and $\frac{r}{s} \cdot \frac{t}{u} = \frac{rt}{su}$, for $\frac{r}{s}, \frac{t}{u} \in R_S$ [1] and (1971) [2]. R is called a regular ring (in the sense of Von Neumann), if for each element $a \in R$, there exists $b \in R$ such that $a = aba$. R is called a Noetherian (an Artinian) ring, if it satisfies the ascending (descending) chain condition for ideals and it is called locally Noetherian, if R_P is Noetherian for each prime ideal P of R . The Jacobson radical of R , denoted by $rad(R)$ (or $J(R)$), is defined as the intersection of all maximal ideals of R (1991) [3] and the prime radical of R , denoted by $Rad(R)$, is defined as the intersection of all prime ideals of R (1970) [4]. An ideal A of R is called finitely generated, if there exists a finite set in R that generates A , that is $A = \langle r_1, r_2, \dots, r_n \rangle$, for $r_i \in R$ and a positive integer n . R is called a local ring if it has only one maximal ideal and it is called a semilocal ring if it has a finite number of maximal ideals. The dimension of R , denoted by $dimR$, is defined as the supremum of all lengths of chains of prime ideals (1974) [5]. It is obvious that, $dimR = 0$ if and only if every prime ideal of R is maximal.

The following are some known theorems on which we depend to prove the results of the present paper.

Let R be a commutative ring with identity. If every prime ideal of R is maximal, then every non zero divisor of R is a unit and if the zero ideal of R is a product of a finite number of maximal ideals then R is Noetherian if and only if R is Artinian (1970) [4]. Furthermore, R is Artinian if and only if R is Noetherian

and $\dim R = 0$ (1970) [4]. If P is a prime ideal of R , then R_P is a local ring with P_P as its unique maximal ideal (1971) [2]. If A is a finitely generated ideal of R such that $AJ(R) = A$, then $A = 0$ [Nakayama Lemma]. Also we have the following equivalent conditions for R (1979) [6].

- (1) R is a Von Neumann regular ring.
- (2) R_P is a field for each maximal ideal P of R .
- (3) R contains no non zero nilpotent elements and every prime ideal of R is maximal.
- (4) All simple R –modules are injective.

Many authors, in [7,8,9,10,11], have studied locally Noetherian rings and almost Noetherian domains and they determined some of their properties. The aim of this paper is first, to study locally Artinian rings and second, to classify those locally Noetherian rings which are locally Artinian.

2. The Main Results

Definition 2.1. Let R be a commutative ring with identity. We say R is locally Artinian if R_P is Artinian for each prime ideal P of R .

It is known that, if R is an Artinian ring and S is a multiplicative system in R , then R_S is Artinian. In fact, if $A'_1 \supseteq A'_2 \supseteq \dots \supseteq A'_n \supseteq \dots$ is any descending chain of ideals in R_S , then for a fixed $s \in S$, the mapping $\rho: R \rightarrow R_S$, defined by $\rho(a) = \frac{sa}{s}$, for $a \in R$, is a homomorphism. Then, we can get the following descending chain of ideals in R , $\rho^{-1}(A'_1) \supseteq \rho^{-1}(A'_2) \supseteq \dots \supseteq \rho^{-1}(A'_n) \supseteq \dots$ and as R is Artinian, there exists a positive integer k such that $\rho^{-1}(A'_k) = \rho^{-1}(A'_{k+1}) = \rho^{-1}(A'_{k+2}) = \dots$, from which one can easily get that $A'_k = A'_{k+1} = A'_{k+2} = \dots$. Hence R_S is Artinian, so that if P is any prime ideal of R , then $S = R \setminus P$ is a multiplicative system in R , so that we get R_P is Artinian. Hence, we get the following corollary.

Corollary 2.2. Every Artinian ring R is locally Artinian.

We mention that, a locally Artinian ring need not be Artinian, in general as we see in the following example..

Example. Consider the set $N = \{1,2,3, \dots\}$ of all natural numbers and let $R = \{A: A \subseteq N\} = P(N)$. Define $(+)$ and $(.)$ as follows: $A + B = (A \cup B) \setminus (A \cap B)$ and $AB = A \cap B$, for $A, B \in R$. One can easily show that $(R, +, .)$ is a commutative ring with N as its identity, \emptyset as its zero and the additive inverse of any $A \in R$ is A itself. First, we claim that R is locally Artinian. Let P be any prime ideal of R . If $A \in R$ is any element, then we have $A.A.A = A$. Hence, R is a Von Neumann regular ring, so that P is a maximal ideal of R and that R_P is a field and hence an Artinian ring, thus R is locally Artinian. Let $F = \{A: A \in R \text{ and } A \text{ is a finite subset of } N\}$. Clearly, $\emptyset \in F$ and thus $\emptyset \neq F \subseteq R$. Now, if $A, B \in F$ and $X \in R$, then $A - B = A + (-B) = A + B$ is finite and hence in F . Also, we have $AX = A \cap X \subseteq A$ and $XA = X \cap A \subseteq A$ so AX and XA are finite subsets of N , thus we get $AX, XA \in F$. Hence F is an ideal of R . We claim that F is not finitely generated. If possible, suppose that F is finitely generated, so that $F = \langle B_1, B_2, \dots, B_n \rangle$, for $B_i \in F$, and $1 \leq i \leq n$. Since each B_i is a finite subset of N , so $B = \cup_{i=1}^n B_i$ is also a finite subset of N . Let $m = \sup \{a: a \in B\}$. Now, let $Y = \{m + 1\}$, which is a finite subset of N and hence $Y \in F$, so there exists $C_1, C_2, \dots, C_n \in R$ such that $Y = C_1 B_1 + C_2 B_2 + \dots + C_n B_n = C_1 \cap B_1 + C_2 \cap B_2 + \dots + C_n \cap B_n \subseteq \cup_{i=1}^n (C_i \cap B_i) \subseteq \cup_{i=1}^n B_i = B$. Hence we get $m + 1 \in B$ and this contradicts the fact that $m = \sup \{a: a \in B\}$, so that F is not finitely generated and that R is not Noetherian. Hence, R is not Artinian.

Theorem 2.3. A locally Artinian ring which is semilocal, is Artinian. (In particular, a locally Artinian ring which is local is Artinian).

Proof. Let P_1, P_2, \dots, P_k be the maximal ideals of R and $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n \supseteq \dots$ be any descending chain of ideals in R . As R is locally Artinian, we get that R_{P_i} is Artinian, for each i . Now, for each i ($1 \leq i \leq k$), we get the descending chain $(A_1)_{P_i} \supseteq (A_2)_{P_i} \supseteq \dots \supseteq (A_n)_{P_i} \supseteq \dots$ of ideals in R_{P_i} . Thus, for each i ($1 \leq i \leq k$), there exist a positive integer m_i such that $(A_{m_i})_{P_i} = (A_{m_i+1})_{P_i} = (A_{m_i+2})_{P_i} = \dots$. Let $m = \max \{m_i: 1 \leq i \leq k\}$, then, clearly for each i ($1 \leq i \leq k$), we get $(A_m)_{P_i} = (A_{m+1})_{P_i} = (A_{m+2})_{P_i} = \dots$. Hence, by proposition 3.13 (1971) [2], we get that $A_m = A_{m+1} = A_{m+2} = \dots$. Hence R is Artinian.

It is known that in an Artinian ring, every prime ideal is maximal, by Theorem 11.9 (1970) [4] and now, we extend this result to locally Artinian rings.

Theorem 2.4. If R is a locally Artinian ring, then every prime ideal of R is maximal.

Proof. Let P be any prime ideal of R . As P is proper, there exists a maximal ideal Q of R , such that $P \subseteq Q$. Then, by Corollary 3.11 (1971) [2], we get P_Q is a prime ideal of R_Q and as R_Q is Artinian, we have P_Q is a maximal ideal of R_Q . But Q_Q is the unique maximal ideal of R_Q , so we get $P_Q = Q_Q$. Now, let $P \neq Q$, so there exists $x \in Q$ and $x \notin P$. Then $\frac{x}{1} \in Q_Q$ and if $\frac{x}{1} \in P_Q$, then $px \in P$, for some $p \notin Q$. As $P \subseteq Q$, we have $p \notin P$ and as P is prime, we get $x \in P$, which is a contradiction and thus $\frac{x}{1} \notin P_Q$, that means $P_Q \neq Q_Q$, which is a contradiction, so that $P = Q$. Hence, P is a maximal ideal of R .

It is known that, the Jacobson radical and the prime radical of an Artinian ring are identical (1991) [3], now we extend this result to locally Artinian rings.

Corollary 2.5. If R is a locally Artinian ring, then $J(R) = Rad(R)$.

Proof. The proof is obvious.

Another property of Artinian ring is that an Artinian domain (Artinian ring which is also an integral domain) is a field, Now, we extend this result to locally Artinian rings.

Theorem 2.6. Any locally Artinian domain R is a field.

Proof. Let P be any maximal ideal of R . It is enough to show that $P = 0$. Since R is locally Artinian, so R_P is Artinian and as R is an integral domain, one can easily get that R_P is also an integral domain and so that R_P is a field. But, we have R_P is a local ring with P_P as its unique maximal ideal, so we must have $P_P = 0$. Next, let $x \in P$, then $\frac{x}{1} \in P_P = 0$, thus there exists $y \notin P$ such that $yx = 0$ and as $y \neq 0$, we must have $x = 0$, that means $P = 0$. Hence R is a field.

Next, we introduce the following useful property.

Definition 2.7 (Primal Completeness Property). We say that a ring R possesses the Primal Completeness Property (pc –property), if the complement of each prime ideal of R consists of no non zero divisors of R .

Now, we support this definition by the following examples.

(1) Every integral domain possesses the pc –property. Let R be an integral domain and P be any prime ideal of R . If $R \setminus P$ contains some zero divisor, say x , then there exists $0 \neq y \in R$ such that $xy = 0$ and as R is an integral domain, we get $x = 0 \in P$, which is a contradiction. Hence, R possesses the pc –property.

(2) In the ring Z_6 , we have $P = \{0, 2, 4\}$ is a maximal ideal of Z_6 and hence a prime ideal of Z_6 . Then $Z_6 \setminus P = \{1, 3, 5\}$ and $3 \in Z_6 \setminus P$ is such that $2.3 = 0$, that means 3 is a zero divisor of Z_6 and thus Z_6 does not possess the pc –property.

In the next result, we prove that locally Artinian rings which possess the pc –property, are Artinian.

Theorem 2.8. A locally Artinian ring R which, possesses the pc –property, is an Artinian ring.

Proof. As R is locally Artinian, by Theorem 2.4, we have every prime ideal of R is maximal. We claim that R has only one prime ideal. Let P_1 and P_2 be any two prime ideals of R , then they are maximal. Let $x \notin P_1$. Since R possesses the pc –property, so x is a non zero divisor of R and since every prime ideal of R is maximal, so we get that x is a unit. Conversely, if x is a unit, then clearly $x \notin P_1$. Hence $R \setminus P_1 = \{x: x \text{ is a unit of } R\}$. By using the same technique, we get that $R \setminus P_2 = \{x: x \text{ is a unit of } R\}$. Hence, we get $R \setminus P_1 = R \setminus P_2$, which gives that, $P_1 = P_2$, that means, R has only one prime ideal, so that R is a local ring and hence by Theorem 2.3, we get R is Artinian.

In fact, the above condition is not necessary, in general, since the ring Z_6 is Artinian but it does not satisfy the pc –property as we have seen in example (2).

It is known that, every Artinian ring is Noetherian, so that every locally Artinian ring is locally Noetherian and the converse is not true, in general. In fact, the ring of integers Z is Noetherian, so it is locally Noetherian and since, the zero ideal of Z is prime but not maximal, so Z is not locally Artinian.

Now, we give a condition under which every locally Noetherian ring is a locally Artinian ring.

Theorem 2.9. Let R be a locally Noetherian ring. If for each prime ideal P of R there exists a positive integer n for which $P^n = P^{n+1}$, then R is locally Artinian.

Proof. Let P be a prime ideal of R , so there exists a positive integer n such that $P^n = P^{n+1}$. Then we get $(P^n)_P = (P^{n+1})_P$ and since P_P is the unique maximal ideal of R_P , so we have $J(R_P) = P_P$. Hence, we get $(P^n)_P = (P^{n+1})_P = (P^n P)_P = (P^n)_P P_P = (P^n)_P J(R_P)$. Now, as R is locally Noetherian, we have R_P is Noetherian and thus $(P^n)_P$ is finitely generated and then by Nakayama's Lemma, we get $(P^n)_P = 0$, that is, $(P_P)^n = 0$, which means that the zero ideal of the Noetherian ring R_P is a product of a finite number of maximal ideals and thus we get R_P is Artinian. Hence R is locally Artinian.

Next, we give two examples, the first does not satisfy the condition that given in Theorem 2.9, while the second satisfies the given condition.

(1) The ring Z is Noetherian and hence it is locally Noetherian and the ideal $\langle 2 \rangle$ is a prime ideal of Z . But there does not exist a positive integer k such that $\langle 2 \rangle^k = \langle 2 \rangle^{k+1}$, since if $\langle 2 \rangle^k = \langle 2 \rangle^{k+1}$, for some positive integer k , then $\langle 2^k \rangle = \langle 2^{k+1} \rangle$. Hence $2^k \in \langle 2^{k+1} \rangle$ and so that $2^k = m2^{k+1}$, for some $m \in Z$. Then, we get $2^k(2m - 1) = 0$ and as Z is an integral domain and $2^k \neq 0$, we get $2m - 1 = 0$, which is a contradiction. Thus, there is no positive integer k such that $\langle 2 \rangle^k = \langle 2 \rangle^{k+1}$.

(2) The ring Z_6 is Noetherian (since it has only a finite number of ideals), so it is locally Noetherian. The only prime ideals of Z_6 are $\{0,2,4\}$ and $\{0,3\}$. Clearly, we have $\{0,2,4\}^2 = \{0,2,4\}$ and $\{0,3\}^2 = \{0,3\}$.

Lemma 2.10. If R is an Artinian ring, then for each ideal I of R there exists a positive integer k such that $I^k = I^{k+1}$.

Proof. The proof is obvious.

Next, in the following theorem we classify those locally Noetherian rings which are locally Artinian.

Theorem 2.11. Let R be a locally Noetherian ring which possesses the $(pc - \text{property})$, then the following conditions are equivalent:

- (1) R is locally Artinian.
- (2) For each prime ideal P of R , there exists a positive integer k such that $P^k = P^{k+1}$.

Proof. (1) \Rightarrow (2) Let R be locally Artinian and P be any prime ideal of R . Since R possesses the $(pc - \text{property})$, so by Theorem 2.8, we get R is an Artinian ring and then by Lemma 2.10, there exists a positive integer k such that $P^k = P^{k+1}$.

(2) \Rightarrow (1) Suppose that the condition (2) holds. Since R is locally Noetherian, then by Theorem 2.9, we get that R is locally Artinian.

As a corollary to the above we give the following.

Corollary 2.12. Let R be a commutative ring with identity which possesses the $pc - \text{property}$, then the following conditions are equivalent:

- (1) R is locally Artinian.
- (2) R is locally Noetherian and for each prime ideal P of R , there exists a positive integer k such that $P^k = P^{k+1}$.

Proof. The proof follows directly from Theorem 2.11.

Lemma 2.13. Let R be a commutative ring with identity. Then $\dim R = 0$ if and only if $\dim R_P = 0$, for each prime ideal P of R .

Proof. (\Rightarrow) Let $\dim R = 0$ and P be any prime ideal of R . To show $\dim R_P = 0$. So, let \bar{P} be any prime ideal of R_P . Then, by Corollary 3.11 (1971) [2], there exists a prime ideal Q of R such that $Q \cap (R \setminus P) = \emptyset$ and $\bar{P} = Q_P$. Then, we get $Q \subseteq P$. As, $\dim R = 0$, we have every prime ideal of R is maximal, thus Q is a maximal ideal of R and since $P \neq R$, we get $Q = P$. Hence, $\bar{P} = Q_P = P_P$ and as P_P is the unique maximal ideal of R_P , we have \bar{P} is maximal. Hence, we get that $\dim R_P = 0$.

(\Leftarrow) Let $\dim R_P = 0$, for each prime ideal P of R . If P is any prime ideal of R , then there exists a maximal ideal Q of R such that $P \subseteq Q$ and then $P_Q \subseteq Q_Q$. Then P_Q is a prime ideal of R_Q and since $\dim R_Q = 0$, so we get P_Q is a maximal ideal of R_Q and thus $P_Q = Q_Q$. If $P \neq Q$, then there exists $x \in Q$ and $x \notin P$, then

$\frac{x}{1} \in Q_Q$. If $\frac{x}{1} \in P_Q$, then $qx \in P$, for some $q \notin Q$, which implies that $q \notin P$ and as P is prime, we get $x \in P$, which is a contradiction. Hence $\frac{x}{1} \notin P_Q$, so that $P_Q \neq Q_Q$, which is again a contradiction. Hence we must have $P = Q$, that means P is a maximal ideal of R . Hence, $\dim R = 0$.

It is known that, a ring R is Artinian if and only if R is Noetherian and $\dim R = 0$ (1970) [4]. Now, we extend this result to locally Artinian rings.

Theorem 2.14. Let R be a commutative ring with identity. Then R is locally Artinian if and only if R is locally Noetherian and $\dim R = 0$.

Proof. (\Rightarrow) Let R be locally Artinian, then it is locally Noetherian and by Theorem 2.4, we get that every prime ideal of R is maximal, so that $\dim R = 0$.

(\Leftarrow) Let R be locally Noetherian and $\dim R = 0$. If P is any prime ideal of R , then, R_P is Noetherian and as $\dim R = 0$, by Lemma 2.13, we have $\dim R_P = 0$, so by Theorem 12.16 (1970) [4], we get R_P is Artinian. Hence R is locally Artinian.

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